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Short Communication

## A note on “a dual-ascent approach to the fixed-charge capacitated network design problem”

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### Abstract

We show by a counterexample that the dual-ascent procedure proposed by Herrmann, Ioannou, Minis and Proth in a 1996 issue of the *European Journal of Operational Research* is incorrect in the sense that it does not generate a valid lower bound to the optimal value of fixed-charge capacitated network design problems. We provide a correct dual-ascent procedure based on the same ideas and we give an interpretation of it in terms of a simple Lagrangean relaxation. Although correct, this procedure is not effective, as in general, it provides a less tighter bound than the linear programming relaxation. © 2002 Elsevier Science B.V. All rights reserved.

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In this paper, we consider the fixed-charge multicommodity capacitated network design problem (MCFP), which can be described as follows. We denote by  $G = (N, E, K)$  an undirected network, where  $N$  is the set of nodes,  $E$  is the set of edges and  $K$  is the set of commodities or origin-destination pairs. For each commodity  $k$ , we denote by  $d^k$  the positive demand that must flow between the origin  $O(k)$  and the destination  $D(k)$ . To each edge  $\{i, j\}$ , we associate a positive capacity  $u_{ij}$ , and we assume that  $u_{ij} \leq \sum_{k \in K} d^k$ . A nonnegative fixed cost  $f_{ij}$  is charged when edge

$\{i, j\}$  is used. Although the network is undirected, the flow is directed and we introduce the set of directed arcs  $A$  corresponding to  $E$ , i.e., for each  $\{i, j\} \in E$ , we define the arcs  $(i, j) \in A$  and  $(j, i) \in A$ . A nonnegative transportation cost  $c_{ij}^k$  has to be paid for each unit of commodity  $k$  moving through arc  $(i, j) \in A$ . The problem consists in minimizing the sum of all costs, while satisfying demand requirements and capacity constraints.

MCFP can be tackled by using mixed-integer programming (MIP) techniques (for surveys on MIP techniques applied to this problem and other related network design problems, see Magnanti and Wong [11], Minoux [12], and Gendron et al. [6]). In particular, some efforts have been devoted

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to design efficient solution techniques for MCFP based on Lagrangean relaxation [3–6,9]. Although in general, Lagrangean relaxation might provide tighter lower bounds than the linear programming (LP) relaxation, Gendron and Crainic [4] have shown that for most “reasonable” Lagrangean relaxations of MCFP, the best Lagrangean bound is equal to the LP bound.

In a paper published in 1996, Herrmann et al. [8], proposed a dual-ascent method, hereafter denoted HIMP, for MCFP. They claim that the bounds obtained by their procedure can improve over the LP bounds for some instances. In this note, we show with a counterexample that their method is incorrect, as in general it does not provide a valid lower bound on the optimal value of the problem (therefore, Theorem 4.6 in [8] is incorrect). Exploiting the same ideas as in the HIMP procedure, we provide a correct dual-ascent procedure and give an interpretation of it in terms of a simple Lagrangean relaxation. Although correct, this procedure is not effective, as in general, it provides a less tighter lower bound than the LP relaxation.

We first recall the formulation of MCFP. We define nonnegative flow variables  $x_{ij}^k$ , which represent the fraction of the flow of commodity  $k$  on arc  $(i, j) \in A$ , i.e.,  $d^k x_{ij}^k$  is the flow of commodity  $k$  on arc  $(i, j)$ . We also introduce binary design variables  $y_{ij}$ , which assume value 1 whenever edge  $\{i, j\} \in E$  is used and value 0 otherwise. The problem is then formulated as follows:

$$Z(\text{MCFP}) = \min \sum_{k \in K} \sum_{(i,j) \in A} d^k c_{ij}^k x_{ij}^k + \sum_{\{i,j\} \in E} f_{ij} y_{ij}, \tag{1}$$

$$\sum_{j \in N} x_{ij}^k - \sum_{j \in N} x_{ji}^k = \begin{cases} 1 & \text{if } i = \text{O}(k), \\ -1 & \text{if } i = \text{D}(k), \\ 0 & \text{if } i \neq \text{O}(k), \text{D}(k), \end{cases} \quad \forall i \in N, k \in K, \tag{2}$$

$$\sum_{k \in K} d^k (x_{ij}^k + x_{ji}^k) \leq u_{ij} y_{ij} \quad \forall \{i, j\} \in E, \tag{3}$$

$$x_{ij}^k, x_{ji}^k \leq y_{ij} \quad \forall \{i, j\} \in E, k \in K, \tag{4}$$

$$x_{ij}^k \geq 0 \quad \forall (i, j) \in A, k \in K, \tag{5}$$

$$y_{ij} \in \{0, 1\} \quad \forall \{i, j\} \in E. \tag{6}$$

Note that an equivalent MIP formulation is defined in [8] which, however, provides a weaker LP relaxation. In this formulation, the capacity constraints (3) are not written as “forcing” constraints as here, i.e., the design variables are not used in the expression of the constraints. This simple difference between the two formulations can have a major impact on the values of the corresponding LP bounds.

The HIMP procedure can be summarized as follows. It is based on relaxing the capacity constraints (3), thus obtaining the formulation of a multicommodity uncapacitated fixed-charge network design problem (MUFP). The procedure applies the dual-ascent method of Balakrishnan et al. [2], hereafter denoted the BMW procedure, on the resulting formulation. At termination, the BMW procedure identifies a subset of “interesting” arcs, which we denote  $A_0$  (see details below). The HIMP procedure then determines if  $A_0$  has enough capacity to satisfy the demands of all commodities. In the affirmative, the heuristic terminates after solving the LP relaxation of MCFP restricted to  $A_0$ , therefore deriving a feasible solution. Otherwise, the algorithm selects a commodity  $k$  and an arc  $(i, j) \in A_0$  such that the demand for commodity  $k$  is blocked by the capacity of edge  $\{i, j\}$  (taking into account all other commodities). The procedure then increases by  $\theta_{ij}^k > 0$ , the corresponding transportation cost. Using this updated transportation cost, the BMW procedure is called again, starting from the solution obtained by the previous execution of the procedure.

In order to fully understand the procedure and the counterexample that follows, it is necessary to recall the details of the BMW procedure, an efficient heuristic that attempts to solve the LP dual of MUFP, based on a labeling method. If we let  $\pi_i^k$  and  $\beta_{ij}^k, \beta_{ji}^k$  denote the dual variables associated to constraints (2) and (4), respectively, the LP dual of MUFP, denoted DMUFP, can be written as follows:

$$Z(\text{DMUFP}) = \max \sum_{k \in K} \pi_{\text{D}(k)}^k, \tag{7}$$

$$\pi_j^k - \pi_i^k - \beta_{ij}^k \leq d^k c_{ij}^k \quad \forall (i, j) \in A, k \in K, \tag{8}$$

$$\sum_{k \in K} (\beta_{ij}^k + \beta_{ji}^k) \leq f_{ij} \quad \forall \{i, j\} \in E, \tag{9}$$

$$\beta_{ij}^k \geq 0 \quad \forall (i, j) \in A, k \in K. \tag{10}$$

To describe the BMW procedure formally, we introduce the following notation. Let  $\bar{c}_{ij}^k$  and  $\bar{f}_{ij}$  denote the slack associated to each constraint (8) and (9), respectively, of DMUFP:

$$\bar{c}_{ij}^k = d^k c_{ij}^k + \beta_{ij}^k - \pi_j^k + \pi_i^k \quad \forall (i, j) \in A, k \in K,$$

$$\bar{f}_{ij} = f_{ij} - \sum_{k \in K} (\beta_{ij}^k + \beta_{ji}^k) \quad \forall \{i, j\} \in E.$$

Also let  $N_U(k)$  denote the set of unlabeled nodes for each commodity  $k$  and associated with this set, define the cutset

$$A(N_U(k)) = \{(i, j) \in A \mid i \in N_U(k), j \in N - N_U(k)\}.$$

Finally, let  $K_U$  be the set of commodities which are still candidates for ascent iterations. The BMW procedure can then be summarized as follows:

1. *Initialization.*

$$\pi_i^k \leftarrow \text{shortest distance from } O(k) \text{ to } i \quad \forall i \in N, k \in K \text{ (using } d^k c_{ij}^k \text{ as arc costs);}$$

$$\bar{f}_{ij} \leftarrow f_{ij} \quad \forall \{i, j\} \in E;$$

$$\bar{c}_{ij}^k \leftarrow d^k c_{ij}^k - \pi_j^k + \pi_i^k \quad \forall (i, j) \in A, k \in K;$$

$$Z(\text{DMUFP}) \leftarrow \sum_{k \in K} \pi_{D(k)}^k.$$

2. *Initialization of unlabeled nodes.*

$$N_U(k) \leftarrow N - \{D(k)\} \quad \forall k \in K;$$

$$K_U \leftarrow K.$$

3. *Selection.* Select a commodity  $k \in K_U$ .

4. *Ascent iteration.*

$$\delta_1 \leftarrow \min\{\bar{f}_{ij} \mid \bar{c}_{ij}^k = 0, (i, j) \in A(N_U(k))\};$$

$$\delta_2 \leftarrow \min\{\bar{c}_{ij}^k \mid \bar{c}_{ij}^k > 0, (i, j) \in A(N_U(k))\};$$

$$\delta \leftarrow \min\{\delta_1, \delta_2\};$$

$$\bar{f}_{ij} \leftarrow \bar{f}_{ij} - \delta \quad \forall (i, j) \in \{(i, j) \in A(N_U(k)) \mid \bar{c}_{ij}^k = 0\};$$

$$\bar{c}_{ij}^k \leftarrow \bar{c}_{ij}^k - \delta \quad \forall (i, j) \in \{(i, j) \in A(N_U(k)) \mid \bar{c}_{ij}^k > 0\};$$

$$\pi_i^k \leftarrow \pi_i^k + \delta \quad \forall i \in N - N_U(k);$$

$$Z(\text{DMUFP}) \leftarrow Z(\text{DMUFP}) + \delta.$$

5. *Labeling.* If  $\delta = \delta_1$ , select an arc

$$(i^*, j^*) \in (i, j) \in A(N_U(k)) \mid \bar{c}_{ij}^k = 0\}$$

such that

$$\bar{f}_{i^* j^*} = 0$$

and set

$$N_U(k) \leftarrow N_U(k) - \{i^*\}.$$

6. *Loop over commodities.* Remove commodity  $k$  from  $K_U$ ; if  $K_U \neq \emptyset$ , go to step 3.

7. *Stopping rule.*

$$K_U \leftarrow \{k \in K \mid O(k) \in N_U(k)\};$$

if  $K_U = \emptyset$ , STOP, otherwise go to step 3.

At the end of the procedure, we have at hand a subset of “interesting” arcs, defined as  $A_0 = \{(i, j) \in A \mid \bar{f}_{ij} = 0\}$ . This set is used by the HIMP procedure, which can be summarized as follows:

1. *Initialization.* Apply the initialization (step 1) of procedure BMW

$$Z^* \leftarrow Z(\text{DMUFP}).$$

2. *Procedure BMW.* Apply procedure BMW, starting from step 2 to compute  $Z(\text{DMUFP})$  and to identify  $A_0$ .

3. *Feasibility check.* Determine if  $A_0$  has enough capacity to satisfy the demands of all commodities; in the affirmative, go to 5.

4. *Variable cost update.* Select a commodity  $k$  and an arc  $(i, j) \in A_0$  such that the demand of commodity  $k$  is blocked by the capacity of edge  $\{i, j\}$ ; determine a positive penalty  $\theta_{ij}^k$ ;  $\bar{c}_{ij}^k \leftarrow \bar{c}_{ij}^k + \theta_{ij}^k$ ;  $Z^* \leftarrow Z(\text{DMUFP}) + \theta_{ij}^k$ ; go to 2.

5. *Termination.*

$$Z^* \leftarrow \max\{Z^*, Z(\text{DMUFP})\};$$

solve the LP relaxation of MCFP restricted to  $A_0$  to derive a feasible solution.

The authors claim that if in step 4, the penalty is chosen as  $\theta_{ij}^k = \min\{\bar{f}_{ij} \mid \bar{f}_{ij} > 0\}$ ,  $Z^*$  always represents a lower bound on  $Z(\text{MCFP})$ . The following example shows, however, that this claim is not true.

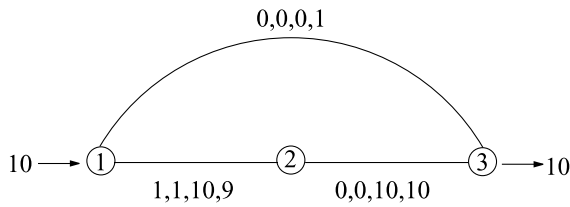


Fig. 1. Network of example.

**Example.** Consider the network of Fig. 1, where a demand of 10 units for one commodity must flow from node 1 to node 3. For each edge  $\{i, j\}$ , the figure indicates the transportation costs  $c_{ij}$ ,  $c_{ji}$ , the fixed cost  $f_{ij}$  and the capacity  $u_{ij}$ . This problem has only one feasible solution, namely,  $x_{13} = 1/10$ ,  $x_{12} = 9/10$  and  $x_{23} = 9/10$  ( $x_{ij} = 0$  on every other arc  $(i, j)$ ), the cost of which is  $Z(\text{DMCFP}) = 29$ . After the initialization, we obtain  $\pi_1 = 0$ ,  $\pi_2 = 10$ ,  $\pi_3 = 0$  and  $Z(\text{DMUFP}) = 0$ . After one iteration of procedure BMW, the origin is labeled and we obtain the same values as after the initialization. The set  $A_0$  consists only of arc  $(1, 3)$ . As it is impossible to send 10 units of flow through this arc, the transportation cost of arc  $(1, 3)$  is increased by  $\theta_{13} = \min\{\bar{f}_{ij} | \bar{f}_{ij} > 0\} = 10$  and  $Z^* = 10$ . We then reapply procedure BMW. After one iteration,  $\pi_3$  has been updated to 10,  $Z(\text{DMUFP}) = 10$  and no node has been labeled. At the second iteration, the origin is labeled (nothing else is modified). Again,  $A_0 = \{(1, 3)\}$  and the transportation cost of arc  $(1, 3)$  is increased by 10, with  $Z^* = 20$ . In the first iteration of the subsequent call to procedure BMW,  $\bar{f}_{23}$  is decreased by 10; now,  $\bar{f}_{23} = 0$ ,  $Z(\text{DMUFP}) = 20$  and node 2 is labeled. In the second iteration, the origin is labeled and  $A_0 = \{(1, 3), (2, 3)\}$ . The penalty on the transportation cost of arc  $(1, 3)$  is increased by 10, since  $(1, 3)$  is the only arc for which flow is blocked by the capacity. But now  $Z^* = 30 > 29 = Z(\text{MCFP})$ , so that  $Z^*$  is not a lower bound on  $Z(\text{MCFP})$ . When procedure BMW is applied again, it labels the origin from arc  $(1, 2)$ , so that  $A_0 = \{(1, 3), (2, 3), (1, 2)\}$  and the procedure finds the only feasible solution.

To understand why the HIMP procedure does not work properly, consider the following relaxation LR of MCFP, where  $b_{ij}^k = \min\{u_{ij}, d^k\}$ :

$$Z(\text{LR}) = \min \sum_{k \in K} \sum_{(i,j) \in A} (d^k c_{ij}^k + \theta_{ij}^k) x_{ij}^k + \sum_{\{i,j\} \in E} f_{ij} y_{ij} - \sum_{k \in K} \sum_{(i,j) \in A} (b_{ij}^k / d^k) \theta_{ij}^k \quad (11)$$

subject to constraints (2) and (4) to (6). This relaxation is obtained from MCFP by: (1) removing the capacity constraints (3); (2) adding the valid inequalities  $x_{ij}^k \leq b_{ij}^k / d^k \forall (i, j) \in A, k \in K$ ; (3) relaxing the latter in a Lagrangean fashion. This relaxation allows us to interpret the HIMP procedure in a simple way. Basically, the procedure solves the LP dual of LR by the BMW procedure and then updates the multipliers  $\theta$ . We only have to slightly modify step 4 of the HIMP procedure to allow it to compute a valid lower bound based on relaxation LR, as the update of  $Z^*$  should now be  $Z^* \leftarrow Z(\text{DMUFP}) + \theta_{ij}^k (1 - b_{ij}^k / d^k)$ . It is easy to verify that when we apply this new version of the procedure on our example, we successively obtain  $Z^* = 9$ ,  $Z^* = 18$  and  $Z^* = 27$ , before the procedure identifies the feasible solution.

Although correct, this revised procedure is not effective. Indeed, it is easy to verify that it provides a less tighter bound than the LP relaxation, since the latter is stronger than relaxation LR. This is true in particular for our example, since the optimal LP solution is:  $x_{12} = 9/10$ ,  $x_{23} = 9/10$ ,  $x_{13} = 1/10$ ,  $y_{12} = 1$ ,  $y_{23} = 9/10$  and  $y_{13} = 1$ , with a cost of  $28 > 27$ .

More effective dual-ascent procedures for MCFP have been studied by Balakrishnan [1]. Guignard and Spielberg [7] have also presented dual-ascent methods for the capacitated plant location problem, a particular case of the MCFP. More recently, Crainic et al. [3] presented dual-ascent procedures based on solving Lagrangean duals with a general bundle method [10].

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